Tutorial 8

Resilient Communication Systems
Prof. Dr.-Ing. Vahid Jamali
Tutor: Mohamadreza Delbari
Email: mohamadreza.delbari@rcs.tu-darmstadt.de

Released: December 1, 2025.

TECHNISCHE UNIVERSITÄT DARMSTADT

RESILIENT COMMUNICATION SYSTEMS

A Fundamental (Diversity-Multiplexing) Trade-off in MIMO Channels

Definition

1. $f(\gamma)\doteq\gamma^b$ if $\lim\limits_{\gamma\to+\infty}\frac{\log f(\gamma)}{\log\gamma}=b$, and $\dot{\geq}$ and $\dot{\leq}$ are similarly defined.
2. $\mathcal{A}'=\mathcal{A}\cap\mathbb{R}^{n+}$ where $\mathcal{A}=\{\boldsymbol{\alpha}:\sum_i(1-\alpha_i)^+<r\}$ and $\mathbb{R}^{n+}$ is the set of real $n-$vectors with non-negative elements. Also $x^+=\max\{0,x\}$, r is data rate and $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_N)$.

Recall

Single antenna channel: Consider $y=\sqrt{\gamma}hx+n$ where $h\in\mathbb{C}$ is Rayleigh channel, and $y,x,n\in\mathbb{C}$. If we set the input data rate to $R=r\log\gamma$ the outage probability is $P_{\text{out}}(r\log\gamma)=P(\log(1+\gamma|h|^2)\leq r\log\gamma)\approx P(|h|^2\leq\gamma^{-(1-r)})\doteq\gamma^{-(1-r)}$, where we know $|h|^2$ is exponentially distributed.

Problem

Consider a point-to-point multiple-input multiple-output (MIMO) system with M transmit antennas and N receive antennas. The channel input-output relation reads

$$\mathbf{Y}=\sqrt{\frac{\gamma}{M}}\mathbf{H}\mathbf{X}+\mathbf{N},$$

where $\mathbf{X}\in\mathbb{C}^{M\times T}$, $\mathbf{Y}\in\mathbb{C}^{N\times T}$, and $\mathbf{H}\in\mathbb{C}^{N\times M}$ are the transmit symbols, the received symbols, and the channel matrix with independently complex circular symmetric Gaussian with unit variance, respectively, and $\mathbf{N}\in\mathbb{C}^{N\times T}$ is a spatially-white AWGN where each entry of $\mathbf{N}$ has mean zero and variance 1. γ is the average SNR at each receive antenna. Assume that $\mathbf{H}$ entries are known to the receiver but not to the transmitter. Assume also $\mathbf{H}$ remains constant within a T period and $T\geq M+N-1$. Let data rate be $R=r\log\gamma$ (b/s/Hz):

1. Based on *Hint 1*, prove that

$$P_{\text{out}}(r\log\gamma)\doteq\int_{\mathcal{A}'}\prod_{i=1}^{N}\gamma^{-(2i-1+M-N)\alpha_i}\,d\boldsymbol{\alpha}. \tag{1}$$

2. Consider $r\leq\min(M,N)$. Based on the result of the previous part, prove that the outage probability satisfies

$$P_{\text{out}}(r\log\gamma)\doteq\gamma^{-d_{\text{out}}(r)}, \tag{2}$$

where

$$d_{\text{out}}(r)=\inf_{\boldsymbol{\alpha}\in\mathcal{A}'}\sum_{i=1}^{\min\{M,N\}}2i-1+|M-N|\alpha_i. \tag{3}$$

3. Based on *Hint 2* and the result of part 2, prove that the probability of a detection error is lower-bounded by
$$P_e(\gamma)\dot{\geq}\gamma^{-d_{\text{out}}(r)}. \quad (4)$$

4. Part 3 provides a lower bound on the error probability. In this part, we will find an upper bound. Consider $P(e|\text{outage}) \approx 1$, via a union bound, derive an upper-bound for
$$P_e(\gamma) = P_{\text{out}}(R)P(e|\text{outage}) + P(e, \text{no outage}), \quad (5)$$
and conclude that $P_e(\gamma)\dot{\leq}\gamma^{-d_{\text{out}}(r)}$.

5. Based on the result of parts 3 and 4 and also *Hint 3* conclude:
$$d(r) = (M - r)(N - r), \quad (6)$$
where d and r are diversity and multiplexing gains, respectively.

Hints

1. Let $\mathbf{H}$ be an $M \times N$ random matrix with i.i.d $\mathcal{CN}(0, 1)$ entries. Suppose $M \geq N$, $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_N$ be the ordered nonzero eigenvalues of $\mathbf{H}^{\mathsf{H}}\mathbf{H}$, then the joint pdf of μ_i's is:
$$p(\boldsymbol{\mu}) = c\prod_{i=1}^{N}\mu_i^{M-N}\prod_{i<j}(\mu_i - \mu_j)^2 \exp\left(-\sum_i \mu_i\right),$$
where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)$. With replacing $\mu_i = \gamma^{-\alpha_i}$, it can be derived:
$$p(\boldsymbol{\alpha}) = c(\log\gamma)^N\prod_{i=1}^{N}\gamma^{-(M-N+1)\alpha_i}\prod_{i<j}(\gamma^{-\alpha_i} - \gamma^{-\alpha_j})^2 \exp\left(-\sum_{i=1}^{N}\gamma^{-\alpha_i}\right), \quad (7)$$
where c is a normalizing constant.

2. Fano's inequality:
$$P(e|\mathbf{H} = \bar{\mathbf{H}}) \geq \frac{RT - 1 - I(\mathbf{X}; \mathbf{Y}|\mathbf{H} = \bar{\mathbf{H}})}{RT}, \quad (8)$$
where it has been conditioned on a specific channel realization $\mathbf{H} = \bar{\mathbf{H}}$ and $I(\cdot;\cdot)$, $P(e)$ show mutual information and probability of error detection, respectively.

3. Let's define
$$\mathbb{V}_k = \{\mathbf{H} \in \mathbb{C}^{N\times M} : \text{rank}(\mathbf{H}) = k\}, \quad (9a)$$
$$\mathbb{U}_k = \{\mathbf{H} \in \mathbb{C}^{N\times M} : \text{rank}(\mathbf{H}) \leq k\}. \quad (9b)$$
We say $\mathbf{H}_k$ is near singular if it is close to the boundary $\mathbb{U}_{k-1}$. In other words, $\mathbf{H}_k$ is near singular when its component in $d_k - d_{k-1}$ dimensions is small where d_k is the dimensionality of the set of $\mathbb{V}_k$.

References

[1] A. Goldsmith, *Wireless communications*. Cambridge university press, 2005.

[2] D. Tse and P. Viswanath, *Fundamentals of wireless communication*. Cambridge university press, 2005.

[3] L. Zheng and D. N. C. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels," *IEEE Transactions on information theory*, vol. 49, no. 5, pp. 1073–1096, 2003.

Lizhong Zheng

Lizhong Zheng is a distinguished Chinese-American electrical engineer, educator, and researcher, renowned for his contributions to information theory and wireless communication. Born in China, Zheng displayed exceptional talent in mathematics and engineering from a young age. He pursued his undergraduate and master's studies in Electronic Engineering at Tsinghua University, earning his B.S. and M.S. degrees in 1994 and 1997, respectively. He then advanced his education at the University of California, Berkeley, where he earned his Ph.D. in Electrical Engineering and Computer Sciences in 2002.

Lizhong Zheng

Since 2002, Zheng has been a faculty member at the Massachusetts Institute of Technology (MIT), where he is currently a Professor of Electrical Engineering in the Department of Electrical Engineering and Computer Science. His research interests encompass information theory, statistical inference, communications, and network theory. Among his most influential works is his research on multiple-input, multiple-output (MIMO) systems, which are now widely used in modern communication standards like 5G.

Throughout his career, Zheng has received numerous accolades for his groundbreaking work, including the IEEE Information Theory Society Paper Award in 2003, the NSF CAREER Award in 2004, and the AFOSR Young Investigator Award in 2007. He was named a Fellow of the Institute of Electrical and Electronics Engineers (IEEE) in 2016 for his contributions to the theory of multiple antenna communication. His contributions have been widely cited and remain influential in both academic research and industry applications.

In addition to his research, Zheng is a dedicated educator and mentor, known for inspiring his students to approach complex problems with curiosity and precision. His commitment to advancing the field of electrical engineering has cemented his place as a leading figure in the study of information theory and communication systems.

Lizhong Zheng's innovative contributions continue to influence the evolution of wireless communication, ensuring his legacy as a prominent thought leader in the field.
Read more in Wikipedia and Homepage.